Kummer's quartic surface associated to the Clebsch top (Symmetry and Singularity of Geometric Structures and Differential Equations)

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Abstract. This note deals with the Kummer surface associated to the Clebsch top with Weber's condition. The Clebsch top is an integrable system describing the rotational motion of a rigid body in an ideal fluid, under special conditions. When restricted to a specific symplectic leaf, given through the so-called Weber condition, there is an associated Kummer surface given as a quartic algebraic surface in $\mathbb{CP}^3$ with 16 double points. The explicit conditions of these singular points are given by theoretical computations and are verified by numerical computation through Gröbner bases.

Key words: Integrable systems, Clebsch top, Kummer surface, singular points, Gröbner bases.
MSC(2010): 70E40, 70H06, 14J28, 14J17, 13P10.

1 Introduction

This article deals with the singular points of the Kummer surface appearing in the Clebsch top, a completely integrable Hamiltonian system describing the rotational motion of a rigid body in an ideal fluid.

The rotational motion of the rigid body is described by the Kirchhoff equations, a Hamiltonian system on the Lie-Poisson space $(\mathfrak{so}(3) \ltimes \mathbb{R}^3)^*$. One can restrict the system to an arbitrary symplectic leaf, which is a coadjoint orbit in $(\mathfrak{so}(3) \ltimes \mathbb{R}^3)^*$ equipped with the orbit symplectic form. Under the Clebsch condition, one obtains an additional constant of motion apart from the Hamiltonian as well as the two Casimir functions and hence the restricted system on the coadjoint orbit is completely integrable in the sense of Liouville and hence (the connected component of) a generic common level surface is a two-dimensional torus by Liouville-Arnol'd Theorem, see e.g. [4].

It has been known that the complexification of the common level surface is an Abelian surface which is a double covering of a Kummer surface since the works by Weber [32] on the basis of his work on the Jacobian θ-functions and Kummer's surfaces [31]. In fact, Weber also gave the explicit solutions for the Kirchhoff equations restricted to a specific coadjoint orbit under the Clebsch condition through Jacobian hyperelliptic functions for an algebraic curve of genus two. See also [3].

One considers such a Kummer surface described by an explicit homogeneous quartic equation as given in [32, 3]. It is known that the Kummer surface can be embedded in $\mathbb{CP}^3$ as a singular quartic surface with 16 double (i.e A_1) singular points, see e.g. [6]. Although the Kummer surface is mentioned in [32, 3], the precise positions of the 16 singular points were not given. In the present article, the precise position of 16 A_1 singular points is detected.

By means of the symbolic computations through the Gröbner bases, the above computations are verified. Further, it is shown that the quartic surface admits 16 double points if and only if the Clebsch condition is satisfied.

There are already many researches on the relations of rigid body dynamics to algebraic geometry for example in [1, 25, 2, 7, 5, 15, 23, 27, 30]. On the other hand, there are relatively fewer researches on the relations between the rigid body dynamics and the Kummer surfaces, except for [32, 20, 3, 10] dealing with Clebsch top and [26] concerning the Euler top, i.e. the free rigid body dynamics. The present paper, as well as [13], gives a complementary accounts on the associated Kummer surfaces appearing in Clebsch top under Weber's condition.

As the separation of the variables and the integration in Clebsch top have been investigated well without Weber's condition [20, 22, 29], natural extensions of the present work may be carried out in the general situation without Weber's condition in future works.

2 The Clebsch top

In this section, we briefly describe the Clebsch top. See, e.g. [16, 13] for more details.

We consider the Kirchhoff equations

$$\begin{cases} \dfrac{\mathsf{d}K_1}{\mathsf{d}t} &= \left(\dfrac{1}{I_3}-\dfrac{1}{I_2}\right)K_2K_3+\left(\dfrac{1}{m_3}-\dfrac{1}{m_2}\right)p_2p_3, \\ \dfrac{\mathsf{d}K_2}{\mathsf{d}t} &= \left(\dfrac{1}{I_1}-\dfrac{1}{I_3}\right)K_3K_1+\left(\dfrac{1}{m_1}-\dfrac{1}{m_3}\right)p_3p_1, \\ \dfrac{\mathsf{d}K_3}{\mathsf{d}t} &= \left(\dfrac{1}{I_2}-\dfrac{1}{I_1}\right)K_1K_2+\left(\dfrac{1}{m_2}-\dfrac{1}{m_1}\right)p_1p_2, \end{cases} \tag{2.1}$$

$$\begin{cases} \dfrac{\mathsf{d}p_1}{\mathsf{d}t} &= \dfrac{1}{I_3}p_2K_3-\dfrac{1}{I_2}p_3K_2, \\ \dfrac{\mathsf{d}p_2}{\mathsf{d}t} &= \dfrac{1}{I_1}p_3K_1-\dfrac{1}{I_3}p_1K_3, \\ \dfrac{\mathsf{d}p_3}{\mathsf{d}t} &= \dfrac{1}{I_2}p_1K_2-\dfrac{1}{I_1}p_2K_1, \end{cases} \tag{2.2}$$

where $(\boldsymbol{K},\boldsymbol{p})=(K_1,K_2,K_3,p_1,p_2,p_3)\in\mathbb{R}^3\times\mathbb{R}^3\equiv\mathbb{R}^6$ and $I_1,I_2,I_3,m_1,m_2,m_3\in\mathbb{R}$ are parameters of the dynamics, which do not depend on the time t. The Kirchhoff equations (2.1) and (2.2) are the Hamilton equations for the Hamiltonian

$$H(\boldsymbol{K},\boldsymbol{p})=\frac{1}{2}\left(\sum_{\tau=1}^{3}\frac{K_\tau^2}{I_\tau}+\sum_{\tau=1}^{3}\frac{p_\tau^2}{m_\tau}\right) \tag{2.3}$$

with respect to the Lie-Poisson bracket

$$\{F, G\}(\boldsymbol{K}, \boldsymbol{p}) = \langle \boldsymbol{K}, \nabla_{\boldsymbol{K}} F \times \nabla_{\boldsymbol{K}} G\rangle + \langle \boldsymbol{p}, \nabla_{\boldsymbol{K}} F \times \nabla_{\boldsymbol{p}} G - \nabla_{\boldsymbol{K}} G \times \nabla_{\boldsymbol{p}} F\rangle$$

on $(\mathfrak{so}(3) \ltimes \mathbb{R}^3)^* \equiv \mathbb{R}^6$. Here, $\nabla_{\boldsymbol{K}} F = \left(\frac{\partial F}{\partial K_1}, \frac{\partial F}{\partial K_2}, \frac{\partial F}{\partial K_3}\right)$, $\nabla_{\boldsymbol{p}} F = \left(\frac{\partial F}{\partial p_1}, \frac{\partial F}{\partial p_2}, \frac{\partial F}{\partial p_3}\right)$ for any $F \in \mathcal{C}^\infty(\mathbb{R}^6)$.

By the general theory of Lie-Poisson systems [24, 28], one can restrict the dynamical system of the Kirchhoff equation (2.1) and (2.2) to coadjoint orbits in $(\mathfrak{so}(3) \ltimes \mathbb{R}^3)^*$, which are generically defined as the common level manifolds for the two Casimir functions

$$C_1(\boldsymbol{K}, \boldsymbol{p}) = K_1 p_1 + K_2 p_2 + K_3 p_3, \tag{2.4}$$

$$C_2(\boldsymbol{K}, \boldsymbol{p}) = p_1^2 + p_2^2 + p_3^2. \tag{2.5}$$

Apart from the Hamiltonian H and C_1 and C_2, the Kirchhoff equations (2.1) and (2.2) admit an additional constant of motion

$$L(\boldsymbol{K}, \boldsymbol{p}) = -\left(\frac{p_1^2}{m_2 m_3} + \frac{p_2^2}{m_3 m_1} + \frac{p_3^2}{m_1 m_2}\right) + \frac{K_1^2}{m_1 I_1} + \frac{K_2^2}{m_2 I_2} + \frac{K_3^2}{m_3 I_3}, \tag{2.6}$$

under the Clebsch condition

$$m_1 I_1 (m_2 - m_3) + m_2 I_2 (m_3 - m_1) + m_3 I_3 (m_1 - m_2) = 0, \tag{2.7}$$

which is equivalent to the condition as follows:

$$\exists \nu, \nu' \in \mathbb{R} \text{ s.t. } \forall \tau = 1, 2, 3, \ \frac{1}{m_\tau} = \nu + \frac{\nu' I_\tau}{I_1 I_2 I_3}. \tag{2.8}$$

Following Weber [32], we consider the specific coadjoint orbit $C_1 = 0$, $C_2 = 1$ in this article. The first condition is introduced to simplify the considerations, while the second condition can be assumed without loss of generality. The Liouville-Arnol'd Theorem (see [4]) implies that (the connected component of) the common level surface $H = h$, $L = \ell$, $C_1 = 0$, $C_2 = 1$ is generically a (real) two-dimensional torus. In the next section, we consider the complexification of this common level surface, from which a Kummer's quartic surface is obtained.

3 The Magri-Skrypnyk parameters and the associated Kummer surface

In what follows, we assume all the variables, the parameters, and the geometric settings are complexified and consider them in the holomorphic category. According to [22], we introduce the parameters j_1, j_2, j_3 by $j_\tau = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\nu'}}{I_\tau} - \sqrt{-\nu}\right)$, $\tau = 1, 2, 3$, where ν, ν' are parameters appearing in (2.8).

Then, the two constants of motion H and L in (2.3) and (2.6) can be replaced by the following functions C_3 and C_4:

$$C_3 = \sum_{\tau=1}^{3} \left\{K_\tau^2 + (j_1 + j_2 + j_3 - j_\tau) p_\tau^2\right\}, \quad C_4 = \sum_{\tau=1}^{3} \left(j_\tau K_\tau^2 + \frac{j_1 j_2 j_3}{j_\tau} p_\tau^2\right). \tag{3.1}$$

By a straightforward computation, we see that the two functions C_3 and C_4 Poisson commute. In fact, we can recover the original constants of motion H and L as suitable linear combinations in the form $\lambda C_2 + \lambda' C_3 + \lambda'' C_4$, $\lambda, \lambda', \lambda'' \in \mathbb{C}$.

Instead of the intersection of the four constants of motion $C_1 = 0$, $C_2 = 0$, $H = h$, $L = \ell$, with some constants h, ℓ, we consider the intersection of the four quadrics $C_1 = 0$, $C_2 = 0$, $C_3 = c_3$, $C_4 = c_4$, where $c_3, c_4 \in \mathbb{C}$ are certain constants.

By the elimination of the coordinates p_1, p_2, p_3 from the system of equations $C_1 = 0$, $C_2 = 1$, $C_3 = c_3$, $C_4 = c_4$ and by the substitution $X_\tau / X_0 = k_\tau^2$, $\tau = 1, 2, 3$, we have the following homogeneous irrational equations in X_1, X_2, X_3, X_4:

$$\sqrt{X_1\left(\ell X_4 + d_2 X_2 - X_3\right)} + \sqrt{X_2\left(m X_4 + d_3 X_3 - d_1 X_1\right)} + \sqrt{X_3\left(n X_4 + d_1 X_1 - d_2 X_2\right)} = 0. \tag{3.2}$$

Here, $d_1 = -1/(j_2 - j_3)$, $d_2 = -1/(j_3 - j_1)$, $d_3 = -1/(j_1 - j_2)$,

$$\ell = \frac{j_1^2 - c_3 j_1 + c_4,}{(j_1 - j_2)(j_1 - j_3)}, \quad m = \frac{j_2^2 - c_3 i_2 + c_4,}{(j_2 - j_1)(j_2 - j_3)}, \quad n = \frac{j_3^2 - c_3 i_3 + c_4,}{(j_3 - j_1)(j_3 - j_2)}.$$

Note that

$$\ell + m + n = 1 \tag{3.3}$$

and

$$\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} = 0 \tag{3.4}$$

hold. See [13] for the details of the computations. The equation (3.2) is written in polynomial form as

$$X_1^2 U_1^2 + X_2^2 U_2^2 + X_3^2 U_3^2 - 2X_1 X_2 U_1 U_2 - 2X_2 X_3 U_2 U_3 - 2X_3 X_1 U_3 U_1 = 0, \tag{3.5}$$

where $U_1 = \ell X_4 + d_2 X_2 - d_3 X_3$, $U_2 = m X_4 + d_3 X_3 - d_1 X_1$, $U_3 = n X_4 + d_1 X_1 - d_2 X_2$. The equation (3.5) is a homogeneous quartic equation in X_1, X_2, X_3, X_4 and hence it defines a projectively algebraic surface in $\mathbb{CP}^3$ whose homogeneous coordinates are $(X_1 : X_2 : X_3 : X_4)$.

4 Description of the 16 singular points on the Kummer surface

In this section, we discuss the singular points of the quartic surface $S \subset \mathbb{CP}^3$ defined through (3.5).

First, we show the following proposition, which is stated e.g. in [19, p.1, Chapter 1, §1]. We here give a proof quoting the modern account [9, Chapter 1, §1.2.3] on Plücker's formula on the degree of the dual hypersurface of a hypersurface.

Proposition 4.1. *A quartic surface in $\mathbb{CP}^3$ has at most 16 double points.* ■

Proof. Given a quartic surface $\Sigma \subset \mathbb{CP}^3$ with δ ordinary double points, the degree of the dual surface $\Sigma^\vee \subset \left(\mathbb{CP}^3\right)^\vee$, which is called the class of Σ in [19], can be calculated as

$$\deg \Sigma^\vee = 4 \cdot (4-1)^2 - 2\delta = 36 - 2\delta$$

by Plücker's formula [9, Example 1.2.8]. Note that $\deg \Sigma^\vee$ is in particular even.

Now, we show that $\deg \Sigma^\vee \geq 4$, i.e. $\deg \Sigma^\vee \neq 0, 2$. If $\deg \Sigma^\vee = 0, 2$, however, we would have $\deg (\Sigma^\vee)^\vee \leq 0, 2$, respectively. By Reflexivity Theorem [9, Chapter 1, §2, Theorem 1.2.2] of the dual (hyper)surfaces, we have $(\Sigma^\vee)^\vee = \Sigma$ and hence $\deg (\Sigma^\vee)^\vee = \deg \Sigma = 4$, which is clearly a contradiction.

Thus, we have $36 - 2\delta \geq 4 \iff \delta \leq 16$. $\square$

Remark 4.1. Instead of using the reflexivity of the dual (hyper)surfaces, we can prove Proposition 4.1 by means of a Lefschetz pencil of genus three curves, similarly to [14, pp. 770-771, Chapter VI, §2]. We take a regular point $P \in \Sigma$ and consider a pencil $\{H_\rho\}_{\rho \in \mathbb{CP}^1}$ of planes passing through P each of whose members H_ρ contains at most one singular point of Σ. Then, $\{H_\rho \cap \Sigma\}_{\rho \in \mathbb{CP}^1}$ is a Lefschetz pencil whose generic members are plane quartic curves and hence algebraic curves of genus three. Now, we use the formula (see [14, Chapter 4, §, p. 509, Proposition])

$$\chi(\Sigma) = 2\chi\left(H_\rho \cap \Sigma\right) + T + \delta - N,$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, $H_\rho \cap \Sigma$ is a generic member of the pencil, T is the number of planes in the pencil which are tangent to the surface Σ, and N is the self-intersection number of a generic member $H_\rho \cap \Sigma$ of the pencil. Note that $T + \delta$ stands for the number of singular members in the pencil.

Since the minimal resolution of Σ is a K3 surface and hence has Euler characteristic 24, we have $\chi(\sigma) = 24 - \delta$. As a generic member $H_r \cap \Sigma$ of the pencil is a smooth plane quartic curve and hence is an algebraic curve of genus three, meaning $\chi\left(H_\rho \cap \Sigma\right) = -4$. The number T coincides with the degree $\deg \Sigma^\vee$ of the dual surface $\Sigma^\vee$, because the pencil $\{H_\rho\}_{\rho \in \mathbb{CP}^1}$ induces a line in the dual projective space $\mathbb{CP}^3$ and the tangent members of the pencil correspond to the intersection of the line and the dual surface $\Sigma^\vee$. By the adjunction formula (or the genus formula), we have $3 = \dfrac{N+0}{2} + 1 \iff N = 4$. For the detail of the general formula, see e.g. [6, Chapter II, §11, p. 85, (16)] or [14, Chapter 4, §1, p. 471].

Therefore, we have $24 - \delta = 2 \cdot (-4) + \deg \Sigma^\vee + \delta - 4 \iff \delta = 18 - \dfrac{\deg \Sigma^\vee}{2}$, and hence $\delta \leq 16$, since $\Sigma^\vee \leq 4$, as we have seen in the above proof. ♦

Although Kummer surfaces were introduced in the earliest reference [21], a Kummer surface in the modern context is defined as follows:
We take a complex two-dimensional torus T which can be realized as the unramified quotient of $\mathbb{C}^2$ by a lattice Λ which is group theoretically isomorphic to $\mathbb{Z}^4$: $T \cong \mathbb{C}^2/\Lambda$. Consider the involution $\tilde{\iota} : \mathbb{C}^2 \ni z \mapsto -z \in \mathbb{C}^2$ given through the multiplication by -1 which clearly preserves the lattice Λ. Hence, we have the induced involution $\iota : T \to T$ which has 16 fixed points appearing at each of the half-period points relative to the lattice Λ. The quotient surface T/Λ is a complex surface with 16 rational double points and the minimal resolution of this singular surface is a K3 surface which is usually called a Kummer surface. (See e.g. [6, Chapter V, §16].)

However, we can also obtain Kummer surfaces as singular quartic surfaces in $\mathbb{CP}^3$.

Proposition 4.2. *A quartic surface in $\mathbb{CP}^3$ with 16 double points is a Kummer surface.* ■

Proof. The minimal resolution $\widetilde{S}$ of the quartic surface S with 16 double points has the trivial canonical bundle as we can prove using the adjunction formula [18, Proposition 5.3.11] that the canonical bundle K_S of S is trivial. Then, by a characterization of the rational double points (see e.g. [18, Theorem 7.5.1]), we see that $\widetilde{S}$ has the trivial canonical bundle $\omega_{\widetilde{S}} \cong \mathcal{O}_{\widetilde{S}}$.

By Lefschetz Theorem on Hyperplane Sections [6, Corollary I.20.5], the fundamental group of S is trivial, since $\pi_1(\mathbb{CP}^3) \cong \{e\}$. Since the fundamental group of a complex surface is not changed through a blowing up at a point, we have $\pi_1(\widetilde{S}) \cong \{e\}$. Thus, $\widetilde{S}$ is a K3 surface.

Now, the 16 exceptional curves on $\widetilde{S}$ obtained through the minimal resolution $\widetilde{S} \to S$ form a set of 16 disjoint (-2)-curves and hence, by [6, Proposition VIII.6.1], $\widetilde{S}$ is a Kummer surface. □

We explicitly detect the positions of the 16 singular points of S, which are not mentioned in [3, 32]. Although the method to find the 16 singular points is sketched in Hudson's book [17, Ch. VIII, §55], the precise position of all the singular points on the surface is not given in terms of the precise homogeneous coordinates. Here, we follow Hudson's method to give the explicit homogeneous coordinates for all the 16 singular points.

The method to find the double points is to transform the quartic equation (3.5) into the normal form of A_1 simple singularity: $\xi\eta = \zeta^2$, where ξ, η, ζ are suitable local affine coordinates around the singular point.

The following 14 double points can be found rather easily:

$$\begin{array}{ccc} & X_1 = X_2 = X_3 = 0, & \\ X_2 = X_3 = U_1 = 0, & X_3 = X_1 = U_2 = 0, & X_1 = X_2 = U_3 = 0, \\ X_1 = U_2 = U_3 = 0, & X_2 = U_3 = U_1 = 0, & X_3 = U_1 = U_2 = 0, \\ & U_1 = U_2 = U_3 = 0; & \end{array} \tag{4.1}$$

$$\begin{aligned} X_1 = U_1 &= X_2U_2 - X_3U_3 = 0, \\ X_2 = U_2 &= X_3U_3 - X_1U_1 = 0, \\ X_3 = U_3 &= X_1U_1 - X_2U_2 = 0; \end{aligned} \tag{4.2}$$

Here, the systems of the equations (4.1) give the isolated double points

$$\begin{array}{ccc} (X_1 : X_2 : X_3 : X_4) = & (0:0:0:1), & \\ (1:0:0:0), & (0:1:0:0), & (0:0:1:0), \\ (0 : nd_3 : -md_2 : d_2d_3), & (-nd_3 : 0 : \ell d_1 : d_3d_1), & (md_2 : -\ell d_1 : 0 : d_1d_2), \\ & (1/d_1 : 1/d_2 : 1/d_3 : 0), & \end{array} \tag{4.3}$$

respectively. The type of these singular points can be found through the following presentations of (3.5):

$$\begin{aligned} X_1U_1(X_1U_1 - 2X_2U_2 - 2X_3U_3) + (X_2U_2 - X_3U_3)^2 = 0, \\ X_2U_2(X_2U_2 - 2X_3U_3 - 2X_1U_1) + (X_3U_3 - X_1U_1)^2 = 0, \\ X_3U_3(X_3U_3 - 2X_1U_1 - 2X_2U_2) + (X_1U_1 - X_2U_2)^2 = 0. \end{aligned}$$

Next, we see that the surface S defined through (3.5) has the isolated double points at the points which satisfy the equations (4.2), i.e.

$$\begin{aligned}(X_1 : X_2 : X_3 : X_4) =&(0 : \ell\beta_1 : \ell\gamma_1 : -d_2\beta_1 + d_3\gamma_1),\\ &(\alpha_2 : 0 : \gamma_2 : -d_3\gamma_2 + d_1\alpha_2),\\ &(\alpha_3 : \beta_3 : 0 : -d_1\alpha_3 + d_2\beta_3),\end{aligned} \tag{4.4}$$

where each $(0 : \beta_1 : \gamma_1), (\alpha_2 : 0 : \gamma_2), (\alpha_3 : \beta_3 : 0)$ is one of the two solutions to the respective system of quadratic equations:

$$\begin{aligned}\frac{m}{d_3}\beta_1^2 - \left(\frac{\ell+m}{d_2} + \frac{n+\ell}{d_3}\right)\beta_1\gamma_1 + \frac{n}{d_2}\gamma_1^2 = 0,\\ \frac{n}{d_1}\gamma_2^2 - \left(\frac{m+n}{d_3} + \frac{\ell+m}{d_1}\right)\gamma_2\alpha_2 + \frac{\ell}{d_3}\alpha_2^2 = 0,\\ \frac{\ell}{d_2}\alpha_3^2 - \left(\frac{n+\ell}{d_1} + \frac{m+n}{d_2}\right)\alpha_3\beta_3 + \frac{m}{d_1}\beta_3^2 = 0,\end{aligned} \tag{4.5}$$

To find other two double points of S, we consider the following presentation of the quartic equation (3.5). As can be easily checked, the equation (3.5) can be rewritten as

$$\begin{aligned}&AU_1^2 + 2BU_1 + C = 0\\ \iff\ &(AU_1 + B)^2 = B^2 - AC,\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}A &= X_1^2 + X_2^2 + X_3^2 - 2X_1X_2 - 2X_2X_3 - 2X_3X_1,\\ B &= X_2Z(X_1 - X_2 + X_3) - X_3Y(X_1 + X_2 - X_3),\\ C &= (X_2Z + X_3Y)^2,\\ Y &= U_3 - U_1,\\ Z &= U_1 - U_2.\end{aligned}$$

The discriminant can be computed as

$$B^2 - AC = -4X_1X_2X_3\left(X_1YZ + X_2ZX + X_3XY\right),$$

where we set $X = U_2 - U_3$ with which $X + Y + Z = 0$. The polynomial $\theta := X_1YZ + X_2ZX + X_3XY$ is called in [17] as the equation of the cubic cone. By (4.6), the quartic equation (3.5) is now written as

$$(AU_1 + B)^2 = -4X_1X_2X_3\theta. \tag{4.7}$$

Note that the point $(X_1 : X_2 : X_3 : X_4) = (1/d_1 : 1/d_2 : 1/d_3 : 0)$, where $U_1 = U_2 = U_3 = 0$, the cubic polynomial θ vanishes. See (4.1) and (4.3).

We now consider the condition which permits a linear factor of the cubic polynomial θ. Taking a set of constant coefficients α, β, γ, we assume that the linear form

$$\alpha U_1 + \beta U_2 + \gamma U_3 \tag{4.8}$$

is a linear factor of the cubic polynomial θ, where at least one of α, β, γ is non-zero. Since the quartic equation (3.5) is invariant through the transformation

$$X_1 \mapsto pX_1,\ X_2 \mapsto qX_2,\ X_3 \mapsto rX_3,\ U_1 \mapsto qrU_1,\ U_2 \mapsto rpU_2,\ U_3 \mapsto pqU_3, \tag{4.9}$$

for arbitrary numbers p, q, r. The transformation (4.9) induces the change of the parameters $(\alpha, \beta, \gamma) \mapsto (qr\alpha, rp\beta, pq\gamma)$ and $qr\alpha+rp\beta+pq\gamma = 0$ for suitable p, q, r. Thus, without loss of generality, we can assume that

$$\alpha + \beta + \gamma = 0. \tag{4.10}$$

By (4.8) and (4.10), we have $\beta X = \gamma Z$, $\gamma X = \alpha Z$, $\alpha Y = \beta X$ and hence $\alpha\beta\gamma\theta = 0 \iff (\beta\gamma X_1 + \gamma\alpha X_2 + \alpha\beta X_3)\alpha Y Z = 0$, meaning $\beta\gamma X_1 + \gamma\alpha X_2 + \alpha\beta X_3 = 0$. Here, we used the fact that each two of U_1, U_2, U_3 are not identically the same and hence X, Y, Z are not identically zero.

As a consequence, if the plane $\alpha U_1+\beta U_2+\gamma U_3 = 0$ is a factor of the cubic surface $\theta = 0$, then $\beta\gamma X_1+\gamma\alpha X_2+\alpha\beta X_3 = 0$ holds. In other words, if θ is divisible by $\alpha U_1+\beta U_2+\gamma U_3$, then there exists $\kappa \in \mathbb{C}$ satisfying

$$\beta\gamma X_1 + \gamma\alpha X_2 + \alpha\beta X_3 = \kappa(\alpha U_1 + \beta U_2 + \gamma U_3),$$

i.e. there exists $a, b, c \in \mathbb{C}$ such that $a\alpha = b\beta = c\gamma$ and

$$aX_1 + bX_2 + cX_3 + \alpha U_1 + \beta U_2 + \gamma U_3 = 0 \tag{4.11}$$

holds identically. The linear identity (4.11) can be written as

$$(a - (\beta - \gamma)d_1)X_1 + (b - (\gamma - \alpha)d_2)X_2 + (c - (\alpha - \beta)d_3)X_3 + (\alpha\ell + \beta m + \gamma n)X_4 = 0$$

and hence we have

$$\alpha\ell + \beta m + \gamma n = 0, \quad a = d_1(\beta - \gamma), \quad b = d_2(\gamma - \alpha), \quad c = d_3(\alpha - \beta).$$

As $a\alpha = b\beta = c\gamma$, we have

$$\alpha(\beta - \gamma)d_1 = \beta(\gamma - \alpha)d_2, \tag{4.12}$$

$$\beta(\gamma - \alpha)d_2 = \gamma(\alpha - \beta)d_3, \tag{4.13}$$

$$\gamma(\alpha - \beta)d_3 = \alpha(\beta - \gamma)d_1. \tag{4.14}$$

Clearly, (4.12), (4.13), (4.14) are dependent. For example, (4.14) follows from (4.12), (4.13).

In fact, the equations (4.12), (4.13), (4.14) are equivalent to the following quadratic equations in two of α, β, γ:

$$\frac{\ell}{d_2}\alpha^2 + \left(\frac{n+\ell}{d_1} + \frac{m+n}{d_2}\right)\alpha\beta + \frac{m}{d_1}\beta^2 = 0, \tag{4.15}$$

$$\frac{m}{d_3}\beta^2 + \left(\frac{\ell+m}{d_2} + \frac{n+\ell}{d_3}\right)\beta\gamma + \frac{n}{d_2}\gamma^2 = 0, \tag{4.16}$$

$$\frac{n}{d_1}\gamma^2 + \left(\frac{m+n}{d_3} + \frac{\ell+m}{d_1}\right)\gamma\alpha + \frac{\ell}{d_3}\alpha^2 = 0. \tag{4.17}$$

By means of the relations (3.3), (3.4), and (4.10), however, we can show that the three quadratic equations (4.15), (4.16), (4.17) are equivalent to each other.

The two double points are characterized by the equations

$$aX_1 = \alpha U_1, \quad bX_2 = \beta U_2, \quad cX_3 = \gamma U_3, \tag{4.18}$$

where $(\alpha : \beta : \gamma)$ is a solution of (4.10) and (4.15) (or equivalently (4.16) or (4.17)). By (4.11), we see that $aX_1 + bX_2 + cX_3 = \alpha U_1 + \beta U_2 + \gamma U_3 = 0$ at these points. We can check that (4.18) give points on the surface S, as, multiplying the quartic polynomial in (3.5) by the square of $a\alpha = b\beta = c\gamma$, we have

$$\begin{aligned}
&a^2\alpha^2 X_1U_1 + b^2\beta^2 X_2U_2 + c^2\gamma^2 X_3U_3\\
&\quad - 2ab\alpha\beta X_1X_2U_1U_2 - 2bc\beta\gamma X_2X_3U_2U_3 - 2ca\gamma\alpha X_3X_1U_3U_1\\
&= a^4X_1^4 + b^4X_2^4 + c^4X_3^4 - 2a^2b^2X_1^2X_2^2 - 2b^2c^2X_2^2X_3^2 - 2c^2a^2X_3^2X_1^2\\
&= -(aX_1 + bX_2 + cX_3)(-aX_1 + bX_2 + cX_3)(aX_1 + bX_2 - cX_3)(aX_1 - bX_2 + cX_3) = 0.
\end{aligned}$$

By (4.7), we see that (4.18) represents two double points of the surface S. (Another proof can be found in [19, Chapter 1, §12, p.22].)

To obtain the explicit homogeneous coordinates, we solve the equations (4.18) and we have $(X_1 : X_2 : X_3 : X_4) = (\alpha/d_1 : \beta/d_2 : \gamma/d_3 : 0)$.

To sum up, we have the following theorem.

Theorem 4.3. *The quartic surface S defined through* (3.5) *admits the following 16 A_1 singular points:*

$$\begin{array}{lll}
 & (X_1 : X_2 : X_3 : X_4) = (0:0:0:1), & \\
(1:0:0:0), & (0:1:0:0), & (0:0:1:0),\\
(0 : nd_3 : -md_2 : d_2d_3), & (-nd_3 : 0 : \ell d_1 : d_3d_1), & (md_2 : -\ell d_1 : 0 : d_1d_2),\\
 & (1/d_1 : 1/d_2 : 1/d_3 : 0), & \\
(0 : \ell\beta_1 : \ell\gamma_1 : -d_2\beta_1 + d_3\gamma_1), & (\alpha_2 : 0 : \gamma_2 : -d_3\gamma_2 + d_1\alpha_2), & (\alpha_3 : \beta_3 : 0 : -d_1\alpha_3 + d_2\beta_3),\\
 & (\alpha/d_1 : \beta/d_2 : \gamma/d_3 : 0), &
\end{array}$$

where $(0 : \beta_1 : \gamma_1), (\alpha_2 : 0 : \gamma_2), (\alpha_3 : \beta_3 : 0)$ are the two solutions to the quadratic equations (4.5)*, respectively, and $(\alpha : \beta : \gamma)$ are the two solutions of* (4.10) *and* (4.15) *(or equivalently* (4.16) *or* (4.17)*).* ■

By Proposition 4.1, we see that these 16 singular points gives a maximal number of double points on the quartic surface S. Further, we see that S is in fact a Kummer surface by Proposition 4.2.

The computations of the explicit positions of the singular points on S can numerically be verified by means of the computation with Gröbner bases.

5 Clebsch condition and the number of double points

Let us denote by P the left-hand side of equation (3.5):

$$P = X_1^2U_1^2 - 2\,X_1\,X_2\,U_1\,U_2 - 2\,X_3\,X_1\,U_3\,U_1 + X_2^2U_2^2 - 2\,X_2\,X_3\,U_2\,U_3 + X_3^2U_3^2$$

and put $H = X_1\,X_2\,X_3\,U_1\,U_2\,U_3$.

Lemma 5.1. *Provided $d_2\,d_1 + d_3\,d_1 + d_3\,d_2 = 0$, there exist two double points of P such that $H \neq 0$.* ■

Proof. Consider the double points $\left(\dfrac{\alpha}{d_1} : \dfrac{\beta}{d_2} : \dfrac{\gamma}{d_3} : 0\right)$ in Theorem 4.3 where $(\alpha : \beta : \gamma)$ is one of the two solutions of (4.15) ($\Longleftrightarrow$ (4.16) $\Longleftrightarrow$ (4.17)). Recall that $\alpha + \beta + \gamma = 0$ as mentioned in (4.10). Clearly, we have $P = 0$ at $\left(\dfrac{\alpha}{d_1} : \dfrac{\beta}{d_2} : \dfrac{\gamma}{d_3} : 0\right)$. Generically, α, β, γ are non zero. Further, by (4.15) ($\Longleftrightarrow$ (4.16) $\Longleftrightarrow$ (4.17)), we have $U_\tau = 0 \Longleftrightarrow \dfrac{\ell + m + n}{d_{\tau'} + d_{\tau''}} = 0$, where $\{\tau, \tau', \tau''\} = \{1, 2, 3\}$. This is impossible, since $\ell + m + n = 1$. Thus, we have $\Phi = X_1 X_2 X_3 U_1 U_2 U_3 \neq 0$ at $\left(\dfrac{\alpha}{d_1} : \dfrac{\beta}{d_2} : \dfrac{\gamma}{d_3} : 0\right)$. □

Proposition 5.2. *P has generically 16 double points distinct from the origin if and only if condition $d_2\,d_1 + d_3\,d_1 + d_3\,d_2 = 0$ holds. In the case $d_2\,d_1 + d_3\,d_1 + d_3\,d_2 \neq 0$, the number of such double points falls to 14.* ■

Proof. Let us search double points such that $\Phi \neq 0$ and $d_2\,d_1 + d_3\,d_1 + d_3\,d_2 \neq 0$. We introduce two new variables w and d and consider the polynomial system

$$S_P = \{P, \frac{\partial P}{\partial X_1}, \frac{\partial P}{\partial X_2}, \frac{\partial P}{\partial X_3}, \frac{\partial P}{\partial X_4}, (d_2\,d_1 + d_3\,d_1 + d_3\,d_2)\,d - 1, \Phi\,w - 1\}.$$

If one chooses the monomial ordering (total degree followed by reverse lexicographic; see [8]) $\omega = tdeg(w, d, n, m, l, d_3, d_2, d_1, X_4, X_3, X_2, X_1)$ and compute a Gröbner basis of S_P with ordering ω, it reduces to $\{1\}$. Hence, provided $d_2\,d_1 + d_3\,d_1 + d_3\,d_2 \neq 0$, there are no double points outside $\{\Phi = 0\}$. We finally observe that the 14 points given by (4.3) and (4.4) are on $\{\Phi = 0\}$, while the two points considered in the proof of Lemma 5.1 and appearing when $d_2\,d_1 + d_3\,d_1 + d_3\,d_2 = 0$ are outside $\{\Phi = 0\}$. □

To sum up, we have the following theorem.

Theorem 5.3. *The quartic surface S defined through (3.5) admits the two singular points $(\alpha/_d 1 : \beta/d_2 : \gamma/d_3 : 0)$ where $(\alpha : \beta : \gamma)$ are the two solutions of (4.10) and (4.15) (or equivalently (4.16) or (4.17)), only if the condition (3.4) holds.* ■

Since the condition (3.4) is a consequence of the Magri-Skrypnyk parameters which can be introduced only if we have the Clebsch condition (2.7) ($\Longleftrightarrow$ (2.8)). Thus, we conclude that we obtain a Kummer surface if and only if we have the Clebsch condition (2.7) or equivalently (2.8) for the original parameters.

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